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INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

DYNAMICS OF A VISCOELASTIC COLLISION BETWEEN SMOOTH SURFACES TWO SPHERICAL SOLID BODIES

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ABSTRACT

The dynamics of a viscoelastic collision between smooth surfaces two spherical solid bodies by the application of the "Method of the specific forces" have been given in the article, and the new conception for the definition of the elastic and the viscous forces in the common case of dynamics of a viscoelastic contact is proposed here by the further development of this method. The forces of viscosity and the forces of elasticity can be found by integration of the specific forces acting inside an elementary volume of the contact zone. Also, the derivation of the integral equations of the viscoelastic forces, the equations for pressure in the contact is presented. Work and Energy in the phases of compression and restitution, and at the rolling shear have been derived. Approximate solutions for the differential equations of movement (displacement) by using the method of equivalent work have been derived. Equations for the normal contact stresses have been obtained. Also, equations for kinematic and dynamic parameters of the viscoelastic collision have been obtained in this article. Examples of the comparison of theoretical results and conclusions have been given in the paper.

KEYWORDS: Viscoelastic forces; Method specific forces; Elementary distributed axial loads; Geometry contact area; Dynamic modules; Dissipative energy; Viscoelastic parameters; Method equivalent work.

INTRODUCTION

The objective of this paper the finding of solutions to the problems of a dynamic contact between smooth surfaces two spherical bodies. It is assumed here that the surface of contact is smooth and in this case we are not considering the influence of roughness on the contact forces, and the initial velocities of contact V_{0x} and V_{0y} are less than the effective sound speed in the volume of deformation V_d (Fig.1). Also the influence of adhesive forces has not been considered in this paper.

As we know, the mechanics of an elastic contact problem between two smooth surfaces have been studied yet in the 19-th century by Herts (1882, 1896) and Boussinesq (1885), and then later, for example, it was examined by many others researchers, such as: Bowden and Tabor (1939); Landau and Lifshits (1944); Timoshenko and Goodier (1951); Archard (1957); Galin (1961); Sneddon(1965); Greenwood and Williamson (1966); Johnson, Kendall and Roberts (1971); Derjaguin, Muller and Toporov (1975); Bush, Gibson and Thomas (1975); Tabor (1977); Johnson (1985); Webster and Sayles, 1986; Stronge (2000); Persson, Bucher and Chiaia (2002); Wriggers (2006); Hyun and Robbins (2007). Also a viscoelastic contact between smooth and rough curvilinear surfaces of two solids already have been researched very widely and their results was published in many different manuscripts (Mindlin,1949; Radok. 1957; Hunter,1960; Goldsmith, 1960; Galin, 1961; Lee, 1962; Graham, 1965; Ting, 1966; Greenwood and Williamson, 1966; Simon, 1967; Jonas, 1982; Padovan, Paramadilok, 1984; Johnson, 1985; Brilliantov, 1996; Brilliantov, Spahn, Hertzsch, Poeschel, 1996; Ramírez, Poeschel, Brilliantov and Schwager, 1999; Stronge, 2000; Barber and Ciavarella, 2000; Goloshchapov, 2001, 2003; Laursen, 2002; Dintwa, 2006; Carbone, Lorenz, Persson and Wohlers, 2009; Harrass, Friedrich , Almajid, 2010; Persson, 2010; Cummins, Thornton and Cleary, 2012; Carbone, Putignano, 2013; Popov 2015). In all these researches for a finding of the viscoelastic forces and stresses, the traditional theories and methods usually have been applied. But, in this paper, the novel theoretical and practical principals have been used for finding these forces and stresses.

Let two spherical bodies having the average statistical masses m_2 and m_2, the average statistical radiuses R_1 and R_2, come into contact with the relative velocities of mutual approach V_{0x} and V_{0y} (see Fig. 1). And let the axis X

coincides with the general normal $\vec{n}$ in the initial point of contact *A*, and axis *Y* is directed along the line of maximal approach of the contacting surfaces in the tangential plane *ZAY* (axis *Z* is placed perpendicular to the plane *XAY* and it is not shown here); O_1 and O_2 are centres mass of the bodies and they are the centres of curvature of the contacting surfaces.

As it is seen here, at the initial moment of the time, the colliding bodies come into contact at the initial point ***A*** with coordinates $x = 0$ and $y = 0$, but at the moment of time t the surfaces of the bodies approach to each other on the size x, which also is the relative displacement of the centres of mass of the contacting bodies. Also it is shown here that: x_1 , x_2 are normal deformations of surfaces of bodies; x is the mutual approach between two surfaces by *X*; r is a current radius of the contact area in the plane *XAY*; h_x is the depth of the contact surface.

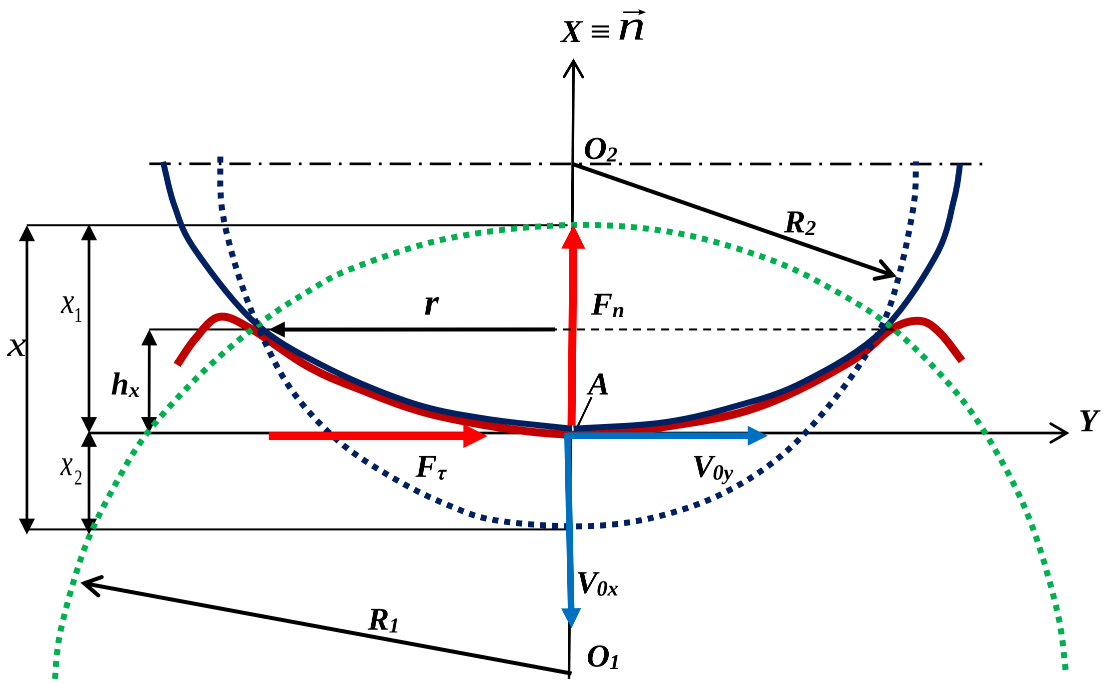

Figure 1. Illustration of the contact between a spherical solid bodies at impact

It is obviously that the bodies are deformed under the influence of the normal viscoelastic force $\overline{F_n}$, the tangential viscoelastic force $\overline{F_\tau}$ and the reactive moment M , and according to Newton's Second Law we can write:

$$\begin{cases} \overline{F_x} = -m_2 \ddot{x} \\ \overline{F_y} = -m_2 \ddot{y} \\ M = -J_z \ddot{\varphi} \end{cases} \qquad (1)$$

Where: m is the effective mass; $\ddot{y}, \ddot{x}$ - the relative accelerations of the centres of mass of the bodies; J_z is the effective moment of inertia of a bodies; φ the relative angle of rotation of the bodies; $\ddot{\varphi}$ is the relative angular acceleration of the bodies; $M = F_\tau l$ is the reactive moment; l is the shoulder of tangential force. As we know, at impact of two bodies, the effective mass m is entered like for the third body, and the movement (the displacement) x of the centre of mass of this third body is taken equal to the distance x the relative displacement of the centres of mass of the colliding bodies. Further in this article, let the third body will be called simply as a body. At impact of two bodies, according to the second law of Newton, we can write that $m\frac{dV_x}{dt} = m_1\frac{dV_{1x}}{dt} = m_2\frac{dV_{2x}}{dt}$, where $\overline{V_x} = \overline{V_{1x}} + \overline{V_{2x}}$, and also we can write that $J_z\frac{d\omega}{dt} = J_1\frac{d\omega_1}{dt} = J_2\frac{d\omega_2}{dt}$, where $\overline{\omega} = \overline{\omega_1} + \overline{\omega_2}$. From these two expressions follows that

$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$, and $\frac{1}{J_z} = \frac{1}{J_{1z}} + \frac{1}{J_{2z}}$.

The viscoelastic forces can be found as the sums of the elastic forces and the viscous forces:

$$\begin{cases} \overline{F_n} = \overline{F_{cn}} + \overline{F_{bn}} \\ F_\tau = \overline{F_{c\tau}} + \overline{F_{b\tau}} \end{cases} \tag{3}$$

Where: $\overline{F_{cn}}$ is the normal elastic force; $\overline{F_{c\tau}}$ is the tangential elastic force; $\overline{F_{bn}}$ is the normal viscous force; $\overline{F_{b\tau}}$ is the tangential viscous force.

Let us to write the equations for elastic forces and viscous forces in the simple form as follows

$$\begin{cases} \overline{F_{cn}} = c_x x, \qquad \overline{F_{bn}} = b_x \dot{x} \\ \overline{F_{c\tau}} = c_y y, \qquad \overline{F_{b\tau}} = b_y \dot{y} \end{cases} \tag{4}$$

Where: x and y are the sizes (distances) of the mutual approach between bodies, which also are the displacements of the centres of mass of the bodies along axes X and Y; $\dot{y}, \dot{x}$ - are the relative velocities of mutual approach between the bodies; b_x is the effective parameter of viscosity at the compression; c_x is the effective parameter of elasticity the compression; b_y is the effective parameter viscosity at the shift; c_y is the effective parameter of elasticity at the shift.

Consequently, according to the equations (1), (2), (3) and (4), we can write the next system of the differential equations of the displacement as:

$$\begin{cases} m\ddot{x} + b_x\dot{x} + c_x x = 0 \\ m\ddot{y} + b_y\dot{y} + c_y y = 0 \\ J_0\ddot{\varphi} + l_x \cdot \left(b_y\dot{y} + c_y y\right) = 0 \end{cases} \tag{5}$$

The most basic problems in the finding of solutions for equations (5) are that, the dynamic contact between two curvilinear surfaces is a non-equilibrium, a nonlinear process of deformations and in this case all mechanical dynamic parameters of viscoelasticity (c_x, c_y, b_x, b_y) are not the constant values. They are variable during of the time of impact, and all dynamic mechanical and physical properties of the materials depending on dynamic conditions of loading and temperature. Therefore, especially for the solving of these problems, such as the definition of the normal viscous force and the all tangential viscoelastic forces, the "Method of the specific forces" have been developed by N. Goloshchapov (Goloshchapov, N., 2015)

In the past many old papers and others published recently (Mindlin, 1949; Simon, 1967; Johnson, 1985 ; Cundall and Strack,1979; Brilliantov, Spahn, Hertzsch, Poeschel, 1996; Schafer, Dippel and Wolf, 1996; Ramírez, Poeschel, Brilliantov, Schwager, 1999; Roylance, 2001; Brilliantov, Poeschel, 2004; Makse, Gland, Johnson and Schwartz, 2004; Schwager and Poschel, Schwager and Poschel, 2007; Becker, Schwagerand, Pöschel, 2008; Schwager and Poschel, 2008; Thornton, 2009; Saitoh, Bodrova, Hayakawa and Brilliantov, 2010; Cummins, Thornton , Cleary, 2012) have been used existing theoretical models, such as the "Linear Spring Dashpot Model"- (LS+D), the "Hertz Mindlin Spring Dashpot Model"- (HM+D), and the "Discrete Elements Method" - (DEM). In all of these methods and models, for the definition of the effective parameter of elasticity c_x (or stiffness, or spring parameter), the Hertz's theory of elastic contact between two surfaces (Landau and Lifshitz, 1944, 1965) has been used. Also for the purpose of finding the tangential forces the coefficient of friction was taken as a constant value. The more comprehensive analysis and review of these already known models and methods can be found, for example, in the monographs of the authors, such as, Stronge, W. J. (2000), Van Zeebroeck, M. (2005), Dintwa, E. (2006), Hongming Li (2006). But, the most basic problems in the finding of solutions for equations (5) are that, the dynamic contact between two curvilinear surfaces is a non-equilibrium, a nonlinear process of deformations and in this case all mechanical dynamic parameters of viscoelasticity (c_x, c_y, b_x, b_y) are not the constant values. They are variable during of the time of impact, and all dynamic mechanical and physical properties of the materials depending on dynamic conditions of loading and temperature. And on other hand, we have to understand that, the Hertz theory allows only the finding the normal elastic force, but it is not enough for finding the viscous normal force and all tangential viscoelastic forces. The existing methods still cannot give the complete answer, how these nonlinear parameters of viscoelasticity can be found for the practical application by using the dynamic modules of elasticity and viscosity, which usually can be found by using the known methods (Ferry, J. D., 1948; Moore, D. F. ,1975; Van Krevelen D. W., 1972; Nilsen, L. E., Landel, R. F., 1994).

Most recent, from already published, researches in the field of the collision of viscoelastic particles (granules) with identical mechanical properties have been made by Brilliantov, Hertzsch, Poeschel, Spahn , Hertzsch, Spahn , Brilliantov (1995, 1996, 2004). They have obtained the equation for the normal viscous force with variable viscosity parameter

$$F_{bn} = F_{dis}^{N} = 4\frac{Y}{(1-\nu^2)}\sqrt{R^{eff}}\,A\sqrt{\xi}\,\xi^{1/2}, \qquad (1^*)$$

where $\xi = x$, $R^{eff} = R$, Y is the Young modulus, ν is the Poisson ratio, $A = \frac{1}{3}\frac{(3\eta_2-\eta_1)^2}{(3\eta_2+2\eta_1)}\left[\frac{(1-\nu^2)(1-2\nu)}{Y\nu^2}\right]$ is the damping viscous parameter, and where η_1 and η_2 are the viscous constants. But this theoretical result can only be used for the researching of the quisistatic contact of the bodies with the same physical-mechanical properties, and in this case we have the problem of finding the viscous constants "η_1" and "η_2". If the contacting surfaces have different physical-mechanical properties this conception does not give the answer, because this is a yet more difficult problem. Also, for finding the equations for the tangential forces again the coefficient of friction was taken as the constant value. But, the coefficient of friction is not a constant value. It can be defined as the follows ratio:

$$f = \frac{F_\tau}{F_n} = \frac{F_{c\tau}+F_{b\tau}}{F_{cn}+F_{bn}} = \frac{c_y y + b_y \dot{y}}{c_x x + b_x \dot{x}} \qquad (6)$$

It is obviously that the coefficient of friction is changing during the period of time of contacting. Thus, as we can see, the many problems still exist now in this research area. Therefore, especially for the solving of these problems, such as the definition of the normal viscous force and the all tangential viscoelastic forces, the "Method of the specific forces" and others theoretical and experimental ways for the finding of the kinematic and the dynamic mechanical parameters between two contacting surfaces, such as the elasticity modulus and the viscosity modulus, have been developed and represented in this article below.

THE METHOD OF THE SPECIFIC FORCES

Let us assume that in the infinitesimal period of the time *dt*, when the mutual approach between the bodies is the infinitesimal magnitude *dx* (Fig. 2), inside the elementary infinitesimal volume *dV*, which is arising around the current point of the contact *A* (Fig.1 and Fig.2), the infinitesimal viscoelastic forces dF_n and dF_τ are beginning to act.
These forces can be found by differentiation of the normal F_{xi} and the tangential F_{yi} specific forces by sizes *da* and *dx*:

$$dF_n = F_{xi}da\,,\; dF_{\tau y} = F_{yi}da + F_{yi}dx \qquad (7)$$

Where: *da* is the diameter of the contact area in the instant of the time *dt*; F_{xi} is the normal effective specific viscoelastic force; F_{yi} is the tangential effective specific viscoelastic force.

According to the "Newton's Third Law" the effective specific forces and the specific forces between contacting bodies have to be equal: $F_{xi} = F_{x1} = F_{x2}$; $F_{yi} = F_{y1} = F_{y2}$. Where: F_{x1} and F_{x2} are the normal specific viscoelastic forces; F_{y1} and F_{y2} are the tangential viscoelastic specific forces. Here and further in this paper the subscript *I =1* is used for a soft body and *I =2* is used for a more rigid body.

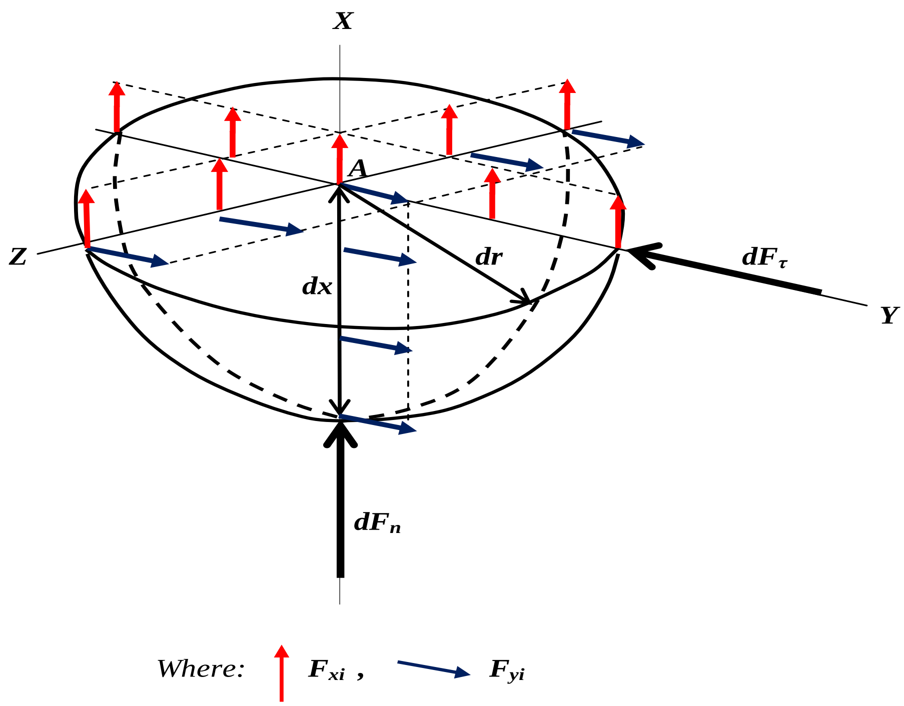

Figure 2. Illustration of the action of the specific viscoelastic forces inside the elementary infinitesimal volume of deformations dV in the vicinity of the current contact point A at the instant of the time dt.

On the other hand, the specific viscoelastic forces can be found as the sum of the specific elastic forces and the specific viscous forces:

$$F_{xi} = F_{xb} + F_{xc}\,,\; F_{x1} = F_{x1b} + F_{x1c}\,,\; F_{yi} = F_{yb} + F_{yc}\,,\; F_{y1} = F_{y1b} + F_{y1c}\,,\; F_{y2} = F_{y2b} + F_{y2c} \qquad (8)$$

Where: F_{xb} is the normal effective specific viscous force; F_{xc} is the normal effective specific elastic force; F_{x1b}, F_{x2b} are the normal specific viscous forces; F_{x1c}, F_{x2c} are the normal specific elastic forces; F_{yb} is the tangential effective specific viscous force; F_{yc} is the tangential effective specific elastic force; F_{y1b}, F_{y2b} are the tangential specific viscous forces; F_{y1c}, F_{y2c} are the tangential specific elastic forces.

THE DEFINITION THE SPECIFIC VISCOELASTIC FORCES AND THE EFFECTIVE DYNAMIC MODULES AND VISCOSITIES BY USING THE "ELEMENTARY DISCRETE ELEMENTS MODEL" (EDEM)

Also let us suppose that the volume of deformation is the system of an infinitely large number of elementary discrete elements (Fig. 3.) connected among themselves definitely. And also, in this case let us assume, that for the infinitesimal period of the contact time *dt* all deformations inside of each elementary discrete element are changing linearly and therefore all specific forces are changing linearly too. Based on this, the equations for all specific forces can be written as the linear functions:

$$\begin{cases} F_{xb} = \eta'_E \dot{x}, \quad F_{xc} = E'x, \quad F_{yb} = \eta'_G \dot{y}, \quad F_{yc} = G'y \\ F_{x1b} = \eta'_{1E}\dot{x}_1, \quad F_{x1c} = E'_1 x_1, \quad F_{y1b} = \eta'_{1G}\dot{y}_1, \quad F_{y1c} = G'_1 y_1 \\ F_{x2b} = \eta'_{2E}\dot{x}_2, \quad F_{x2c} = E'_2 x_2, \quad F_{y2b} = \eta'_{2G}\dot{y}_2, \quad F_{y2c} = G'_2 y_2 \end{cases} \tag{9}$$

Where: E' is the effective dynamic elasticity modulus at the compression; η'_E is the effective dynamic viscosity at the compression; G' is effective dynamic elasticity modulus at the shear; η'_G is the effective dynamic viscosity at the shear; E'_1, E'_2 are the dynamic elasticity modules; η'_{1E}, η'_{2E} are the dynamic viscosities; G'_1, G'_2 are the dynamic elasticity modulus at the shear; η'_{1G}, η'_{2G} are the dynamic viscosity at the shear.

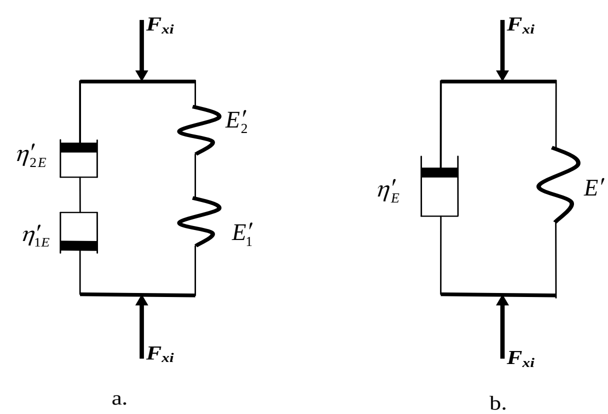

Figure 3. Illustration of the "Elementary discrete elements model (EDEM)": a. the elementary discrete element of the normal contact between the bodies; b. the effective elementary discrete element of the normal contact between the bodies.

It is obvious that the specific elastic forces are equal at the initial instant of the contact, when $t = 0$, $x = 0$ (in this point they are equal zero) and they are equal at the instant of the maximum compression between bodies, when $x = x_m$ (in this point they reach the equal maximum value). But at the same time, the specific viscous forces are equal at the initial instant of the contact, $t = 0$, $x = 0$ (in this point they reach the equal maximum value) and they are equal at the instant of the maximum compression between bodies, $x = x_m$ (in this point they are equal zero). All these forces in the Eq. 9 are linear continuous functions and if they are equal at two values of argument, they have to be equal at any other values also, or by other words, they are equal in any instant of the time of the contact. Thus consequently we can write that

$$F_{xc} = F_{x1c} = F_{x2c}\,,\ \ F_{xb} = F_{x1b} = F_{x2b}\,,\ \ F_{yc} = F_{y1c} = F_{y2c}\,,\ \ F_{yb} = F_{y1b} = F_{y2b} \qquad (10)$$

In the proposed model, each elementary deformation between two bodies develops analogically like the deformation of the elementary discrete element, which is depicted in Fig. 3.a. It is a simple case of the linear model of deformations of elementary discrete elements, and instead this model with four elements we can use its analogy – the model with two effective elements depicted in Fig. 3.b. Also the "Elementary discrete elements model" for the normal forces can be used for the tangential forces in the same manner. Since we have here the case of the linear model of viscoelastic deformation, we can find the effective compliances as the sums of the elastic and the viscous compliances as $\frac{1}{E'} = \frac{1}{E'_1} + \frac{1}{E'_2}$, $\frac{1}{\eta'_E} = \frac{1}{\eta'_{1E}} + \frac{1}{\eta'_{2E}}$, $\frac{1}{G'} = \frac{1}{G'_1} + \frac{1}{G'_2}$, $\frac{1}{\eta'_G} = \frac{1}{\eta'_{1G}} + \frac{1}{\eta'_{2G}}$ and finally the formulas for calculation of the effective dynamic viscosities and the effective dynamic modules of elasticity can be written as:

$$E' = \frac{E'_1 E'_2}{E'_1 + E'_2}\,,\quad \eta'_E = \frac{\eta'_{1E}\eta'_{2E}}{\eta'_{1E} + \eta'_{2E}}\,,\ G' = \frac{G'_1 G'_2}{G'_1 + G'_2}\,,\quad \eta'_G = \frac{\eta'_{1G}\eta'_{2G}}{\eta'_{1G} + \eta'_{2G}} \qquad (11)$$

Now, according to (9), (10) follows that $F_{xc} = E'x = E'_1 x_1 = E'_2 x_2$, and then according to (11) we can write respectively that

$$x_1 = D_1 x \text{ and } x_2 = D_2 x \qquad (12)$$

Where: $D_1 = \frac{E'_2}{E'_1 + E'_2}$ and $D_2 = \frac{E'_1}{E'_1 + E'_2}$ are the coefficients of deformations.

Finding of the integral equations viscoelastic forces

Now, having found all the specific forces and since as the areas of integration *a* and h_x are known (Fig.1 and Fig.2), according to the equations (7), (8) and (9) we can write respectively

$$dF_n = 2F_{xi}da = 2\eta'_E \dot{x}da + 2E'xda\,,\ dF_\tau = F_{yi}da + F_{yi}dx = \eta'_G \dot{y}da + G'yda + \eta'_G \dot{y}dx + G'ydx \qquad (13)$$

In the equations (13) the dynamic viscosities can be replaced at the dynamic viscosity modules according to the known expressions (Ferry, 1963; Van Krevelen , 1972)

$$\frac{E''}{\omega_x} = \eta'_E \text{ and } \frac{G''}{\omega_y} = \eta'_G \qquad (14)$$

Where: E'' is the effective viscosity modulus; G'' is the effective viscosity modulus at shear; ω_x is the frequency of damped oscillations by axis *X*; ω_y is the frequency of damped oscillations by axis *Y*. Viscosity modules can be found by using the known (Ferry, J. D., 1948; Moore, D. F. ,1975; Van Krevelen D. W., 1972; Nilsen, L. E., Landel, R. F., 1994) formula

$$\frac{E''}{E'} = \frac{G''}{G'} = tg\beta\,, \qquad (15)$$

where β is the angle of mechanical losses.

Remark: In dynamics of the continuous environment the dynamic module of elasticity also named as the accumulation or storage modulus, and the dynamic module of viscosity also named as the loss modulus (Ferry, J. D., 1948; Moore, D. F. ,1975; Van Krevelen D. W., 1972; Nilsen, L. E., Landel, R. F., 1994).

According to the Eq.13 the six integral equations can be written respectively as

$$\begin{cases} F_{cn} = 2E'\int xda\,,\ F_{bn} = 2\dfrac{E''}{\omega_x}\dot{x}\int da \\ F_{hb\tau} = \dfrac{G''}{\omega_y}\dot{y}\int_0^{h_x} dx\,,\ F_{hc\tau} = G'y\int_0^{h_x} dx\,,\ F_{ab\tau} = \dfrac{G''}{\omega_y}\dot{y}\int da\,,\ F_{ac\tau} = G'y\int da \end{cases} \qquad (16)$$

Remark: If the area of the contact surface is not a circle, but an ellipse, then we will get eight equations, because in this case we will have two areas of integrating by the big and by the small axes of an ellipse.

CONSIDERATION OF THE GEOMETRY OF THE CONTACT BETWEEN TWO SPHERICAL SURFACES

And now, the important moment, it can be seen that for a finding of the solutions for all these equations (16) we have to know only the equations or the formulas for $r = f(x)$, $a = f(x)$, and for $h_x = f(x)$. For example, we can use that $r = (Rx)^{1/2}$ according to the Hertz theory, but according to this theory, the area of contact is a flat surface and the depth of indentation (the depth of the contact surface) $h_x = 0$. But in reality the area of contact usually is not a flat, it is a curvilinear surface. In Hertz's theoretical models has been taken that the contacting surfaces deform together without of the sliding, but in reality each surface deforms independently. Therefore, to find the radius of the contact area r in reality, let us to consider the geometry of the contact between two spherical surfaces, like it is depicted in the illustrations in Fig.4.

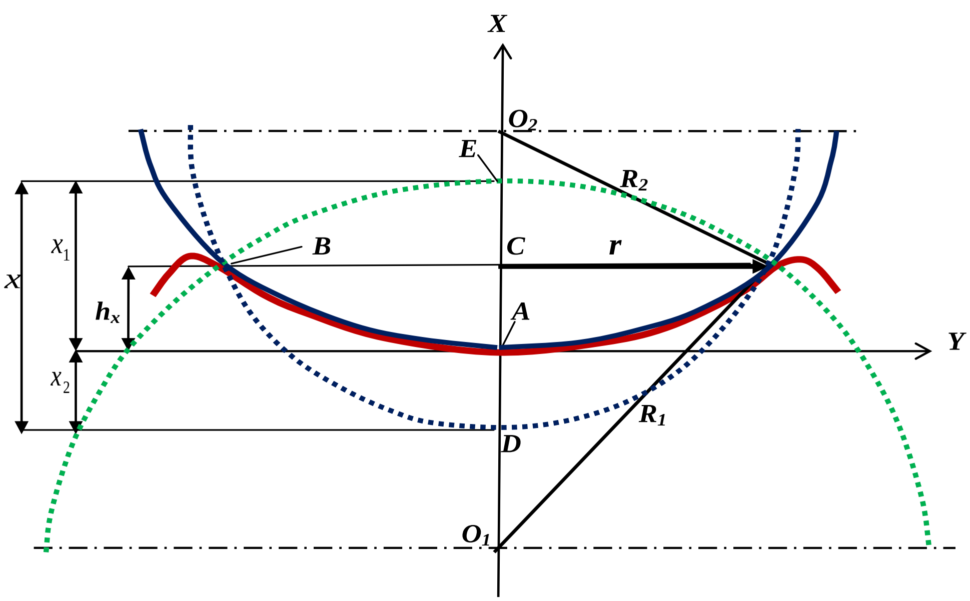

Figure 4. Illustration of the contact between two spherical surfaces

We know that a collision of freely moving bodies is the special state; it is the period of time when the colliding bodies are not affected by any external forces. It is not a compression of two bodies under the influence of the external force when only the certain part of the bodies in the contact zone is deformed. In the initial instant of the time of freely moving collision of two bodies or two particles the Newtonian force of inertia begins to act:

$F_x = -m\ddot{x} = -(\sum_{i=1}^{n} m_i)\ddot{x}$, where m_i is the elementary mass of the body. It is obviously, if the initial speed of impact less than the sound speed inside the volume of deformation, all elementary masses of a body will be involved in the movement together in the same time and all space of a body will be deformed in the same time as well. If a body is elastic or viscoelastic, the position of the centre mass of a body relative to the initial position of the main axes of inertia of the body will not be changed and the magnitudes of the moments of inertia of a body will not changed during the time of a collision, because if they will be changed, the continuity of an environment inside a body will be broken. Also it is obviously that, in the time of indentation of more hard surface into a soft surface, the contact surface takes a curvilinear shape, where the point *B* (see Fig.4) is a special point where the deformations always equal zero, and the border of the area of contact always pass through this point *B*. According to this statement, for example in the case of contact between two spherical bodies (see Fig.5), the distance O_2B between this point and the centre of curvature O_2 of the surface of more hard body will not be changed in the period of time of contact. This distance always equals to the radius of curvature R_2. Also the distance O_1B between this point and the centre of curvature O_1 of the surface of less hard body will not be changed in the period of time of contact too. This distance always equals to the radius of curvature R_1. Hence, obviously that $O_2B = O_2D = R_2$ and $O_1B = O_1E = R_1$, and also we can write that $O_1C + O_2C = (R_1 + R_2) - x$, and since as $O_1C = (R_1^2 - r^2)^{1/2}$ and $O_2C = (R_2^2 - r^2)^{1/2}$, after a simple calculation, if to neglect by members of smallest order, we get the next equation for the radius of contact area $r = f(x)$:

$$r^2 = 2Rx - x^2 \qquad (17)$$

where $R = \dfrac{R_1 R_2}{R_1 + R_2}$ is the effective radius of contact curvature

The equation (17) is not convenient for using and therefore, let us rewrites it as

$$r^2 = k_p^2 Rx, \qquad (18)$$

where $k_p = \sqrt{2 - \dfrac{x}{R}}$ is the correlation coefficient. If a deformation is small, when $R >> x$, hence that $k_p = \sqrt{2}$

Practically for the solution of the contact problems of mechanics, the correlation coefficient can be found by the method of iterations and consecutive approximations.

Obvious from (18) that the surface of the contact takes the parabolic shape

$$x = \frac{1}{k_p^2 R} r^2 \qquad (19)$$

Since the surface of the contact has a parabolic shape, let us to take that the radial distribution of the pressure inside of this area changes analogically according to the parabolic function as

$$P = P_c\left(1 - \frac{r_y^2}{r^2}\right) \qquad (20)$$

Where: r_y is a current radius of the contact area along axis Y; P_c is the maximum magnitude of the pressure in the centre of the contact area.

Further since the square under this function and the square under the linear function of the mean pressure P_m in the contact area are equal, we get

$$P_c\int_0^r\left(1-\frac{r_y^2}{r^2}\right)dr_y = P_m r \ , \tag{21}$$

than after the integration

$$P_c\left(r-\frac{1}{3}r\right)=P_m r \,, \tag{22}$$

and finally the ratio between maximum and the mean pressure in the contact zone can be found as

$$P_c = \frac{3}{2}P_m \tag{23}$$

Now let us to define h_x - the depth of the contact surface (Fig.4 and Fig.1). The expression for the radius of contact area can be found also as follows (Fig.4):

$$r^2 = R_2^2 - \left(R_2^2 - (x_2 + h_x)\right)^2 \tag{24}$$

After a simple geometric calculation, if to neglect by members the smallest order, we obtain the next equation for the radius of contact area:

$$r^2 = 2R_2(x_2 + h_x) \tag{25}$$

Then after the comparison equations (17) and (25) we can write that

$$2R_2(x_2 + h_x) \approx 2Rx \tag{26}$$

Finally since $x_2 = D_2 x$, the formula for h_x can be written as follows

$$h_x = \left(\frac{R - D_2 R_2}{R_2}\right)x = k_h x \ , \tag{27}$$

where $k_h = \left(\frac{R - D_2 R_2}{R_2}\right)$ is the coefficient of the depth of the contact surface.

For example, in the case of contact between a spherical body and a semi-space, when $R_2 = R$ follows that $k_h = (1 - D_2) = D_1$, and hence

$$h_x = x_1 = D_1 x \tag{28}$$

Finding equations for the viscoelastic forces

Now it is necessary to prove that the equations in the system (16) are valid and correct. Since as $da = 2dr$ and we know contact radius from (18), we can find the derivative for a by x

$$da = \frac{k_p R^{1/2}}{x^{1/2}}dx \,, \tag{29}$$

And then after integration of the integral for the elastic force from the system of equations (16) we get

$$F_{cn} = 2k_p E' R^{1/2}\int x^{1/2}dx = \frac{4}{3}k_p E' R^{1/2} x^{3/2} \tag{30}$$

If $k_p = 1$ we have the same solution that has been obtained using the Hertz theory (L. Landau, 1944, 1965). The obtained result is the proof that this method of finding the normal elasticity force definitely is valid and correct, and that the Hertz's theoretical model can be use as the partial case, if taken that the contact surface is a flat. It is obviously now, if we know a functional dependency between *r* and *x*, we can always find the elastic force. But, if this method is valid and correct for definition this force, then it should be valid for the definition of all viscoelastic forces in the equations (16). Thus, if the functional dependency between *r* and *x* is known, for example according to the Hertz theory or it can be found by using a simple geometrical analysis, we as well can find other viscoelastic forces by an integration of the integrals from the system of equations (16) respectively as follows:

$$\begin{cases} F_{bn} = 2\dfrac{k_p E'' R^{1/2}}{\omega_x}\dot{x}(t)\displaystyle\int\dfrac{dx}{x^{1/2}} = 4\dfrac{k_p E''}{\omega_x}R^{1/2}\dot{x}(t)x^{1/2}, \\ F_{hb\tau} = \dfrac{G''}{\omega_y}\dot{y}(t)\displaystyle\int_0^{h_x} dx = \dfrac{G''}{\omega_y}\dot{y}(t)h_x, \\ F_{hc\tau} = G'y(t)\displaystyle\int_0^{h_x} dx = G'y(t)h_x, \\ F_{ab\tau} = \dfrac{k_p G''}{\omega_y}R^{1/2}\dot{y}(t)\displaystyle\int\dfrac{dx}{x^{1/2}} = 2k_p\dfrac{G''}{\omega_y}R^{1/2}\dot{y}(t)x^{1/2}, \\ F_{ac\tau} = k_p G'y(t)R^{1/2}\displaystyle\int\dfrac{dx}{x^{1/2}} = 2k_p G'y(t)R^{1/2}x^{1/2} \end{cases} \tag{31}$$

Remark: Here $\dot{x}(t), \dot{y}(t), y(t)$ are the functions linear independent from *x*, and they cannot be integrated by *x*, and they stay outside of integrals.

Since as h_x is known, the equations for viscous $F_{hb\tau}$ and $F_{hc\tau}$ finally can be written as follows:

$$F_{hb\tau} = \frac{k_h G''}{\omega_y}x\dot{y}, \quad F_{hc\tau} = k_h G'yx \tag{32}$$

After the summation of all partial equations for the elastic and the viscous tangential forces we get the general equations for the tangential forces:

$$F_{b\tau} = \frac{G''}{\omega_y}P_x\dot{y} \quad \text{and} \quad F_{c\tau} = G'P_x y, \tag{33}$$

where

$$P_x = k_h x + 2k_p R^{1/2}x^{1/2} \tag{34}$$

Thus, finally the equations for the normal and tangential viscoelastic forces can be written as

$$\begin{cases} F_n = \dfrac{4k_p E'' R^{1/2}}{\omega_x}\dot{x}x^{1/2} + \dfrac{4}{3}k_p E' R^{1/2}x^{3/2} \\ F_\tau = \dfrac{G''}{\omega_y}P_x\dot{y} + G'P_x y \end{cases} \tag{35}$$

Now according to equations (1), (2), (5) and (35) the system differential equations of the displacement (movement) of the centre of mass of a body can be written as follows:

$$\begin{cases} m\ddot{x} + \dfrac{4k_p E'' R^{1/2}}{\omega_x}\dot{x}x^{1/2} + \dfrac{4}{3}k_p E' R^{1/2} x^{3/2} = 0 \\ m\ddot{y} + \dfrac{G''}{\omega_y}P_x\dot{y} + G'P_x y = 0 \\ J_0\ddot{\varphi} + \left(\dfrac{G''}{\omega_y}P_x\dot{y} + G'P_x y\right)l_x = 0 \end{cases} \quad (36)$$

Thus, the formulas for the variable viscoelasticity parameters in the system of equation (5) can be written respectively as

$$b_x = \frac{4k_p E'' R^{1/2}}{\omega_x}x^{1/2},\ \ c_x = \frac{4}{3}k_p E' R^{1/2} x^{1/2},\ \ b_y = \frac{G''}{\omega_y}P_x,\ \ c_y = G'P_x \quad (37)$$

As we can see, indeed that, the parameters of viscoelasticity are not the constant magnitudes, but they are the functions of the displacement ***x***.

WORK AND ENERGY

As we know the period of time at impact includes two principally different phases such as, the phase of the compression and the phase of the restitution. Also in the duration of a collision, the full initial kinetic energy of the colliding bodies divides into the two independent parts such as, the normal initial kinetic energy $W_x = \dfrac{mV_{0x}^2}{2}$ and the tangential initial kinetic energy $W_y = \dfrac{mV_{0y}^2}{2}$. On the other hand, the full kinetic energy at the instant of rebound $W_t = \dfrac{mV_t^2}{2}$ (where V_t is the relative velocity between the centres of mass of the bodies in the instant of rebound) includes two independent parts such as, the normal kinetic energy at the instant of rebound $W_{tx} = \dfrac{mV_{tx}^2}{2}$ (where V_{tx} is the normal relative velocity between the centres of mass of the bodies in the instant of the rebound) and the tangential kinetic energy the instant of rebound $W_{ty} = \dfrac{mV_{ty}^2}{2}$ (where V_{ty} is the tangential relative velocity between the centres of mass of the bodies in the instant of rebound). Therefore, the description of the processes of the compression and the restitution along the axis *X*, and the shear along the axis *Y* are given independently in this part of the paper.

WORK AND ENERGY IN THE PHASES OF COMPRESSION AND RESTITUTION

The graphical illustration of the functional dependences between the normal viscoelastic forces and the displacement of the centre of mass of the bodies is depicted in Fig. 5: (a). Also the "Rheological model of Kelvin-Vogt", which usually is used for the viscoelastic contact, is represented in Fig. 5: (b). It is obvious that the normal initial kinetic energy W_x is spent for the work A_{xm} of the normal viscoelastic force F_n in the compression phase. But on other hand, A_{xm} can be found as the sum of the works A_{xcm} and A_{xbm}, where A_{xcm} is the work of the normal elastic force F_{cn} and A_{xbm} is the work of the normal viscous force F_{bn} in the compression phase. Also we can say that the part of the kinetic energy W_x is transformed into the potential energy of the nonlinear elastic element (spring) (Fig.5: (b)) and the other part of this kinetic energy is dissipated during the time of deformation at the compression of the nonlinear viscous element (dashpot).

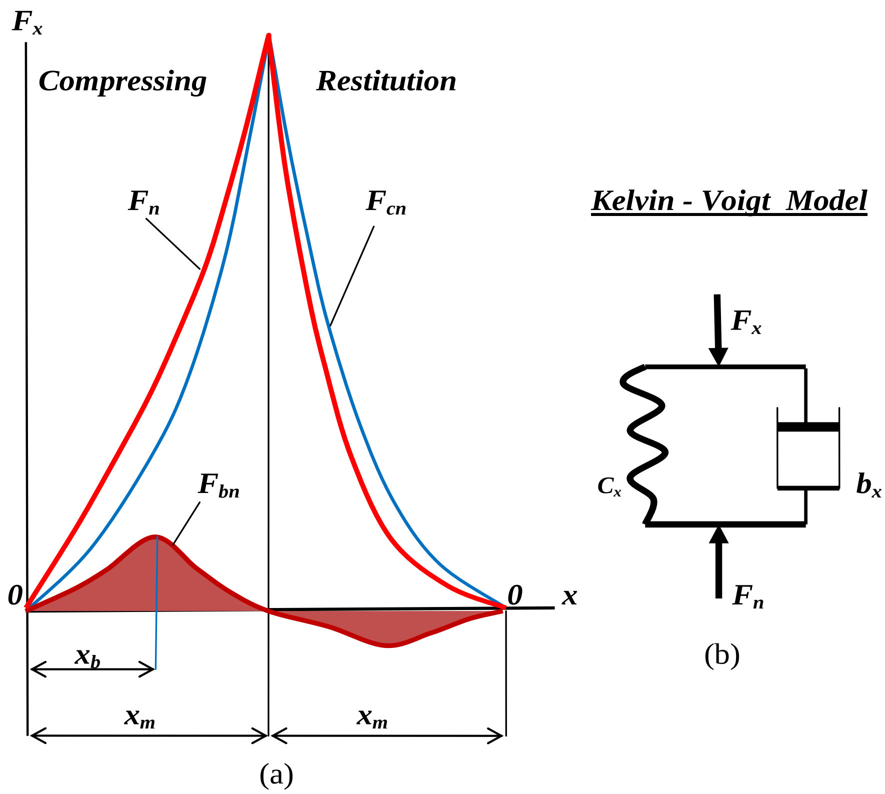

Figure 5: (a) - The graphical illustration of the functional dependences between the normal viscoelastic forces and the displacement x of the centre of mass of the bodies; (b) - The "Nonlinear Rheological Model of Kelvin-Vogt", where c_x and b_x are not the constant magnitudes.

However, on the other hand, the work A_{xt} of the normal viscoelastic force F_n in the restitution phase is equal to the normal energy of the bodies W_{tx} at the instant of rebound, and also A_{xt} can be found as the difference between A_{xct} and A_{xbt}, where A_{xct} is the work of the normal elastic force F_{cn} and A_{xbt} is the work of the normal viscous force F_{bn} in the restitution phase. Consequently, we can write that

$$\begin{cases} A_{xm} = A_{xcm} + A_{xbm} = W_{0x} = \dfrac{mV_{0x}^2}{2} \\ \\ A_{xt} = A_{xct} - A_{xbt} = W_{tx} = \dfrac{mV_{tx}^2}{2} \end{cases} \tag{38}$$

It is obvious that $A_{xcm} = A_{xct}$ and hence the potential energy which has been accumulated inside of the elastic element (spring) fully returns back to the bodies in the instant of rebound. The works A_{xcm} and A_{xbm} at the compression can be found by integration:

$$A_{xcm} = \int_0^{x_m} F_{cn}dx = \frac{4}{3}k_p E'R^{1/2}\int_0^{x_m} x^{3/2}dx = \frac{8}{15}k_p E'R^{1/2}x_m^{5/2} \tag{39}$$

and

$$A_{xbm} = \int_0^{x_m} F_{bn} dx = \int_0^{x_m} \frac{4k_p E'' R^{1/2}}{\omega_x} \dot{x} x^{1/2} dx = \left(\frac{4k_p E'' R^{1/2}}{\omega_x} \right) \frac{\int_0^{x_m} \int dx x^{1/2} dx}{\int_0^{\tau_1} dt} = \frac{8k_p E'' R^{1/2} x_m^{5/2}}{5\omega_x \tau_1} \quad (40)$$

Analogically the works A_{xct} and A_{xbt} in the restitution phase can be found as follows:

$$A_{xct} = -\int_{x_m}^{0} F_{cn} dx = -\int_{x_m}^{0} \frac{4}{3} k_p E' R^{1/2} x^{3/2} dx = \frac{8}{15} k_p E' R^{1/2} x_m^{5/2} \quad (41)$$

and

$$A_{xbt} = -\int_{x_m}^{0} F_{bn} dx = -\int_{x_m}^{0} \frac{4k_p E'' R^{1/2}}{\omega_x} \dot{x} x^{1/2} dx = -\left(\frac{4k_p E'' R^{1/2}}{\omega_x} \right) \frac{\int_{x_m}^{0} x^{1/2} dx \int dx}{\int_{\tau_1}^{\tau_x} dt} = \frac{8k_p E'' R^{1/2} x_m^{5/2}}{5\omega_x \tau_2} \quad (42)$$

Where: $\tau_x = \tau_1 + \tau_2$ is the period time of the contact; τ_1 is the period time of the compression; τ_2 is the period time of the restitution; x_m is the maximum magnitude of the compression between bodies (also it is the maximum displacement of the centre of mass of the bodies, which is equal to the maximum of mutual approach between bodies). According (38), (39), (40), (41) and (42) the equations for the work of the compression and the restitution can be written as follows:

$$\begin{cases} A_{xm} = A_{xcm} + A_{xbm} = \frac{8}{15} k_p R^{1/2} x_m^{5/2} (E' + \frac{3E''}{\omega_x \tau_1}) \\ A_{xt} = A_{xct} - A_{xbt} = \frac{8}{15} k_p R^{1/2} x_m^{5/2} (E' - \frac{3E''}{\omega_x \tau_2}) \end{cases} \quad (43)$$

and according (15) and (43) we can write

$$\begin{cases} A_{xm} = \frac{8}{15} k_p E' R^{1/2} x_m^{5/2} \left(1 + \frac{3tg\beta}{\omega_x \tau_1} \right) \\ A_{xt} = \frac{8}{15} k_p E' R^{1/2} x_m^{5/2} \left(1 - \frac{3tg\beta}{\omega_x \tau_2} \right) \end{cases} \quad (44)$$

Since as $A_{xm} = W_x = \frac{mV_x^2}{2}$, $A_{xt} = W_{tx} = \frac{mV_{tx}^2}{2}$, and by using the first of the equations (44), we get the formula for x_m respectively

$$x_m = \left[\frac{15m\omega_x \tau_1 V_{0x}^2}{16(3tg\beta + \omega_x \tau_1) k_p E' R^{1/2}} \right]^{2/5} \quad (45)$$

ISSN: 2277-9655
(I2OR), Publication Impact Factor: 3.785

Also, we can define the energetic coefficient of restitution e_x, which equals to the square of the kinematic coefficient of restitution k_x (further it will be named simply the coefficient of restitution), like the ratio between W_{tx} and W_{0x}:

$$e_x = k_x^2 = \frac{V_{tx}^2}{V_{0x}^2} = \left(\frac{\omega_x \tau_2 - 3tg\beta}{\omega_x \tau_1 + 3tg\beta}\right)\frac{\tau_1}{\tau_2} \qquad (46)$$

Since as we can take that

$$x_m \approx \frac{/V_{0x}/}{2}\tau_1 = \frac{/V_{tx}/}{2}\tau_2, \qquad (47)$$

we get that

$$k_x = \frac{\tau_1}{\tau_2}, \qquad (48)$$

and using (46) and (48) we get that

$$tg\beta = \frac{\omega_x \tau_1}{3} \times \frac{1-k_x}{k_x} \qquad (49)$$

Thus, we have got the equation, which binds the coefficient of restitution and the tangent of the angle of mechanical losses. So, if $k_x = 1$, $tg\beta \rightarrow 0$ we get the totally elastic impact, but if $k_x = 0$, $tg\beta \rightarrow \infty$ then we get the totally viscous impact. Using (49) we can write the formula for the restitution coefficient as

$$k_x = \left[\frac{\omega_x \tau_1}{(3tg\beta + \omega_x \tau_1)}\right] \qquad (50)$$

If to compare the equations (45) and (50) we can finally get the expression for the maximum magnitude of the compression between a body and a semi-space respectively as

$$x_m = \left[\frac{15mV_{0x}^2}{16k_p E' R^{1/2}} k_x\right]^{2/5} \qquad (51)$$

In the case of a totally elastic impact, when $k_x = 1$ and $k_p = 1$ we get the same result, as it has been obtained by L. Landau (1944, 1965) according to the Hertz theory for the totally elastic contact.

Work and Energy at the rolling shear

It is obvious that, in the during time of the displacement and the rolling shear along axis *Y*, the tangential initial kinetic energy of the bodies W_y is spent for the work A_y of the tangential viscoelastic force $\overline{F_\tau}$. The work A_y can be found as the sum of the works A_{yb} and A_{yc}, where A_{yb} is the work of the tangential viscous force $\overline{F_{b\tau}}$ and A_{yc} is the work of the tangential elastic force. But on other hand, it is obvious as well, that the work A_{yb} is transformed into the dissipative energy Q_ω and the work A_{yc} is transformed into the work A_ω of the relative rotation between bodies. Thus, according to the "Law of the preservation of energy for a non-conservative (dissipative) mechanical systems", we can write equations for the relative displacement of the centres of mass of the bodies and for the relative rotation of the bodies, as follows below:

$$\begin{cases} \frac{m}{2}\left(\frac{dy}{dt}\right)^2 + A_y = \frac{mV_y^2}{2} \\ \frac{J_z}{2}\left(\frac{d\varphi}{dt}\right)^2 + A_\omega + Q_\omega = \frac{J_z\omega_0^2}{2} \end{cases} \quad (52)$$

Where: $A_y = \int F_\tau dy$; $A_\omega = -\int M d\varphi$; $Q_\omega = \int F_{b\tau} dy$, and where $M = F_\tau l$.

Since $F_\tau = F_{c\tau} + F_{b\tau}$ and since as $d\varphi = dy/l$, hence

$$A_\omega = -\int M d\varphi = -\int (F_{c\tau} + F_{b\tau}) dy \quad (53)$$

Also since, if the initial angular velocity ω_0 equals zero we can write the equations (52) for the boundary conditions in the instant of the time $t = \tau_1$ of the maximum compression $x = x_m$ and $y = y_1$ as follows:

$$\begin{cases} \frac{mV_{my}^2}{2} + \int_0^{y_1} (F_{c\tau} + F_{b\tau}) dy = \frac{mV_y^2}{2} \\ \frac{J_z\omega_m^2}{2} - \int_0^{y_1} (F_{c\tau} + F_{b\tau}) dy + \int_0^{y_1} F_{b\tau} dy = 0 \end{cases} \quad (54)$$

Where: V_{my} is the velocity at the instant of the time $t = \tau_1$; ω_m is the relative angular velocity between bodies at the instant of the time $t = \tau_1$; y_1 is displacement of the centres of mass of the bodies along axis Y at the instant of the time $t = \tau_1$. The equations (54) can be rewritten as

$$\begin{cases} \int_0^{y_1} (F_{c\tau} + F_{b\tau}) dy = \frac{mV_y^2}{2} - \frac{mV_{my}^2}{2} \\ \frac{J_z\omega_m^2}{2} = \int_0^{y_1} F_{c\tau} dy \end{cases} \quad (55)$$

Also at the point of the rebound, when $t = \tau_x$ we get

$$\begin{cases} \frac{mV_{ty}^2}{2} + \int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau}) dy = \frac{mV_{my}^2}{2} \\ \frac{J_z\omega_t^2}{2} - \int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau}) dy + \int_{y_1}^{y_t} F_{b\tau} dy = \frac{J_z\omega_m^2}{2} \end{cases} \quad (56)$$

Then we can write that

ISSN: 2277-9655
(I2OR), Publication Impact Factor: 3.785

$$\begin{cases} \dfrac{mV_{ty}^2}{2} + \displaystyle\int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau})dy = \dfrac{mV_{my}^2}{2} \\ \dfrac{J_z\omega_t^2}{2} - \dfrac{J_z\omega_m^2}{2} = \displaystyle\int_{y_1}^{y_t} F_{c\tau}dy \end{cases} \tag{57}$$

The summation of the systems (55) and (57) together yields the following result

$$\begin{cases} \dfrac{mV_{ty}^2}{2} = \dfrac{mV_y^2}{2} - \displaystyle\int_0^{y_1} (F_{c\tau} + F_{b\tau})dy - \int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau})dy \\ \dfrac{J_z\omega_t^2}{2} = \displaystyle\int_0^{y_1} F_{c\tau}dy + \int_{y_1}^{y_t} F_{c\tau}dy \end{cases} \tag{58}$$

We can rewrite equations (58) in the next order

$$\begin{cases} A_y = \dfrac{mV_y^2}{2} - \dfrac{mV_{ty}^2}{2} = \displaystyle\int_0^{y_1} (F_{c\tau} + F_{b\tau})dy + \int_{y_1}^{y_t} (F_{c\tau} + F_{b\tau})dy \\ \dfrac{J_z\omega_t^2}{2} = \displaystyle\int_0^{y_1} F_{c\tau}dy + \int_{y_1}^{y_t} F_{c\tau}dy \end{cases} \tag{59}$$

Finally, we get

$$\begin{cases} A_y = \dfrac{mV_y^2}{2} - \dfrac{mV_{ty}^2}{2} = A_{ybm} + A_{ycm} + A_{ybt} + A_{yct} \\ A_{yc} = \dfrac{J_z\omega_t^2}{2} = A_{ycm} + A_{yct} \end{cases} \tag{60}$$

Where: $A_{ybm} = \displaystyle\int_0^{y_1} F_{b\tau}dy$ is the work of the tangential viscous force $\overline{F_{b\tau}}$ in the compression period τ_1;

$A_{ycm} = \displaystyle\int_0^{y_1} F_{c\tau}dy$ is the work of the tangential elastic force $\overline{F_{c\tau}}$ in the compression period τ_1;

$A_{ybt} = -\int_{y_1}^{y_t} F_{b\tau} dy$ is the work of the tangential viscous force $\overline{F_{b\tau}}$ in the restitution period τ_2;

$A_{yct} = -\int_{y_1}^{y_t} F_{c\tau} dy$ is the work of the tangential elastic force $\overline{F_{c\tau}}$ in the restitution period τ_2.

Since as from Eqs.33 follows that $F_{b\tau} = \frac{G''}{\omega_y} P_x \dot{y}$ and $F_{c\tau} = G' P_x y$, all these works in (60) can be found by integration, as follows:

$$\begin{cases} A_{ybm} = \dfrac{G''}{\omega_y} \displaystyle\int_0^{P_m} dP_x \int_0^{y_1} \dot{y} dy = \dfrac{G''}{\omega_y} P_m \dfrac{\int_0^{y_1} \int dy dy}{\int_0^{\tau_1} dt} = \dfrac{G''}{2\omega_y} P_m \dfrac{y_1^2}{\tau_1} \\ A_{ycm} = G' \displaystyle\int_0^{P_m} dP_x \int_0^{y_1} y dy = \dfrac{G'}{2} P_m y_1^2 \\ A_{ybt} = -\dfrac{G''}{\omega_y} \displaystyle\int_{P_m}^{0} dP_x \int_{y_1}^{y_t} \dot{y} dy = -\dfrac{G''}{\omega_y} (0 - P_m) \dfrac{\int_{y_1}^{y_t} \int dy dy}{\int_{\tau_1}^{\tau_x} dt} = \dfrac{G''}{2\omega_y} P_m \dfrac{y_t^2 - y_1^2}{\tau_x - \tau_1} \\ A_{yct} = -G' \displaystyle\int_{P_m}^{0} dP_x \int_{y_1}^{y_t} y dy = \dfrac{G'}{2} P_m (y_t^2 - y_1^2) \end{cases} \quad (61)$$

Where

$$P_m = k_h x_m + 2k_p R^{1/2} x_m^{1/2} \quad (62)$$

The full changing of the energy of the dissipative system at the rolling shear can be found as the difference between A_y and A_{yc} from (60):

$$\Delta W_y = A_y - A_{cy} = \frac{mV_y^2}{2} - \frac{mV_{ty}^2}{2} - \frac{J_z \omega_t^2}{2} = A_{ybm} + A_{ybt} = A_{yb} \quad (63)$$

According to the equations (60) the conclusion can be drawn that the work A_{yc} is transformed into the kinetic energy of the relative rotation between the bodies, but on the other hand the work A_{yb} is transformed into dissipative energy Q_ω in the process of the internal friction. Accordingly, using (60) and (61) we have

$$A_{yc} = \frac{J_z \omega_t^2}{2} = \frac{G'}{2} P_m y_t^2 \quad (64)$$

Hence, the equation for the relative angular velocity at the instant time of rebound can be written as follows

$$\omega_t = \left(\frac{G' P_m}{J_z} \right)^{\frac{1}{2}} y_t \quad (65)$$

Since the work A_{yb} of the viscous tangential force $F_{b\tau}$ is equal to the dissipative energy Q_ω , using equations (61) we get

$$A_{by} = Q_\omega = A_{ybm} + A_{ybt} = \frac{G''}{2\omega_y} P_m \left(\frac{y_1^2}{\tau_1} - \frac{y_1^2}{\tau_2} + \frac{y_t^2}{\tau_2} \right) \quad (66)$$

Since as $\tau_1 = k_x \tau_2$, finally we get

$$A_{by} = Q_\omega = A_{ybm} + A_{ybt} = \frac{G''}{2\omega_y \tau_2 k_x} P_m \left[y_1^2 (1 - k_x) + k_x y_t^2 \right] \quad (67)$$

APPROXIMATE SOLUTION TO THE DIFFERENTIAL EQUATIONS OF THE DISPLACEMENT BY USING THE METHOD OF THE EQUIVALENT WORKS

For practical application of the differential equations (5) with the variable viscoelasticity parameters, we can find their approximate solutions in the same manner as for the equations with the equivalent constant viscoelasticity parameters, if we choose the equivalent constant parameters B_x, C_x and B_y, C_y so that the work A_{xcm} and A_{xbm} , A_{ycm} and A_{ybm} with the variable viscoelasticity parameters c_x, b_x and c_y, b_y will be equal to the works with the constant viscoelasticity parameters. Thus, according to this statement and according to a boundary value problem, which has to satisfy to the boundary conditions $x = x_m$ and $y = y_1$, and using the known expressions for work A_{xcm} and A_{xbm} , A_{ycm} and A_{ybm} from (39), (40) and (61), we can write next equations

$$\begin{cases} A_{xcm} = C_x \int_0^{x_m} x dx = \frac{1}{2} C_x x_m^2 = \frac{8}{15} k_p E' R^{1/2} x_m^{5/2} \\ A_{xbm} = B_x \int_0^{x_m} \dot{x} dx = B_x \frac{\int_0^{x_m} x dx}{\int_0^{\tau_1} dt} = B_x \frac{x_m^2}{2\tau_1} = \frac{8 k_p E'' R^{1/2}}{5 \Omega_x \tau_1} x_m^{5/2} \end{cases} \quad (68)$$

and also in the phase of the rolling shear for the period of the compression time

$$\begin{cases} A_{ycm} = C_y \int_0^{y_1} y dy = \frac{1}{2} C_y y_1^2 = \frac{1}{2} G' P_m y_1^2 \\ A_{ybm} = B_y \int_0^{y_1} \dot{y} dy = B_y \frac{\int_0^{y_1} \int dy dy}{\int_0^{\tau_1} dt} = B_y \frac{y_1^2}{2\tau_1} = \frac{G''}{2\omega_y \tau_1} P_m y_1^2 \end{cases} \quad (69)$$

Hence, according to the results obtained in (68) and in (69), we can write the expressions for the equivalent constant viscoelasticity parameters, respectively as:

$$B_x = \frac{16 E'' k_p R^{1/2}}{5 \omega_x} x_m^{1/2}, \quad C_x = \frac{16}{15} k_p E' R^{1/2} x_m^{1/2}, \quad B_y = \frac{G''}{\omega_y} P_m, \quad C_y = G' P_m \quad (70)$$

Thus, the equations (5) with variable parameters can be rewritten as the equations with constant parameters as follows:

$$\begin{cases} m\ddot{x} + B_x\dot{x} + C_x x = 0 \\ m\ddot{y} + B_y\dot{y} + C_y y = 0 \\ J_z\ddot{\varphi} + l_x \cdot \left(B_y\dot{y} + C_y y\right) = 0 \end{cases} \tag{71}$$

The equations (71) are the equations of the damped oscillations and the solutions to these equations are known:

$$\begin{cases} x = \dfrac{\upsilon_{0x}}{\omega_x} e^{-\delta_x t} \sin(\omega_x t) \\ \\ y = \dfrac{\upsilon_{0y}}{\omega_y} e^{-\delta_y t} \sin(\omega_y t) \end{cases} \tag{72}$$

Where: $\omega_x = \sqrt{\omega_{0x}^2 - \delta_x^2}$; $\delta_x = \dfrac{B_x}{2m}$ is the normal damping factor; $\omega_{0x} = \sqrt{\dfrac{C_x}{m}}$ is the angular frequency of the harmonic oscillations by axis X ; $\omega_y = \sqrt{\omega_{0y}^2 - \delta_y^2}$; $\delta_y = \dfrac{B_y}{2m}$ is the tangential damping factor; $\omega_{0y} = \sqrt{\dfrac{C_y}{m}}$ is the angular frequency of the harmonic oscillations by axis Y.

It is obviously that the period of time of the contact τ_x is equal to the semi-period of damped oscillations $T_x/2$ by axis X.

$$\tau_x = \frac{T_x}{2} = \frac{\pi}{\omega_x} \tag{73}$$

Since as $\tau_x = \tau_1 + \tau_2$ and also by using equations (46), (48),(49) and (73) we get:

$$tg\beta = \frac{\pi}{3} \times \frac{(1 - k_x)}{(1 + k_x)} \tag{74}$$

The equation for the restitution coefficient we can write now as follows:

$$k_x = \frac{(\pi - 3tg\beta)}{(\pi + 3tg\beta)} \tag{75}$$

If $tg\beta = 0$ hence $k_x = 1$, it is a totally elastic impact, but if $tg\beta = \pi/2$ hence $k_x = 0$ and $x = 0$, it is absolutely plastic impact. Both of these two cases are not possible in nature. Finally, from (51) and (75) follows that

$$x_m = \left[\frac{15mV_{0x}^2}{16k_p E' R^{1/2}} \times \frac{(\pi - 3tg\beta)}{(\pi + 3tg\beta)} \right]^{2/5} \tag{76}$$

Thus we have a very simple way to calculate x_m, if we know the value of $tg\beta$. According to the equations (11), (14) and (15) $tg\beta$ can be calculated by formula

$$tg\beta = \frac{E''}{E'} = \frac{E_1''E_2''(E_1' + E_2')}{E_1'E_2'(E_1'' + E_2'')} \quad (77)$$

The equations for the relative velocities of the centres of mass of the bodies can be received by differentiation of (72):

$$\begin{cases} \dot{x} = \dfrac{V_{0x}}{\omega_x} e^{-\delta_x t}\left[\omega_x \cos(\omega_x t) - \delta_x \sin(\omega_x t)\right] \\ \\ \dot{y} = \dfrac{V_{0y}}{\omega_y} e^{-\delta_y t}\left[\omega_y \cos(\omega_y t) - \delta_y \sin(\omega_y t)\right] \end{cases} \quad (78)$$

Using (78) for the velocity, the duration of the time of the impact equals to the period of the time of the contact can be found now from the conditions $\dot{x} = V_{tx}$ and $t = \tau_x$ as

$$\tau_x = -\frac{\ln k_x}{\delta_x}, \quad (79)$$

where

$$\delta_x = \frac{B_x}{2m} = \frac{8k_p E'' R^{1/2}}{5m\omega_x} x_m^{1/2} = \frac{8k_p E' tg\beta}{5\pi m} \tau_x R^{1/2} x_m^{1/2}, \quad (80)$$

and since $tg\beta$ is known from (74), by using (51),(79) and (80) we get

$$\tau_x^2 = -\frac{2(1+k_x)\ln k_x}{V_{0x}^{2/5}(1-k_x)k_x^{1/5}} \times \left(\frac{5m}{8k_p E' R^{1/2}}\right)^{4/5} \quad (81)$$

DETERMINATION OF THE DYNAMICS MODULES BY THE METHOD OF THE "TEMPERATURE-TIME SUPERPOSITONS"

The dynamic elasticity and viscosity modules for high velocities of the collision can be found, if to follow the principles of the "Time-temperature superposition" according to the equation of the "WLF" Williams - Landel - Ferry or Arrhenius (Ferry, 1963; Van Krevelen , 1972; Moore, 1975; Nilsen and Landel, 1994). First of all we have to define experimentally the effect of temperature for the period of the contact time τ_x, and for the coefficient of restitution k_x at the fixed initial velocity of impact. For example, if we define these parameters for velocity at 2 m/s, then using the principles of the "Time-temperature superposition" we can determine their values for any velocities interesting for us, for example for velocity 100 m/c and for temperature 100 0C. After this, when τ_x and k_x will be known, we can find the value of $tg\beta$ and the dynamic modules E'' and E'. If to use the equation (81), the expression for the calculation of the effective dynamic elasticity module can be written as follows

$$E' = \frac{5m}{8k_p V_{0x}^{1/2} R^{1/2} \tau_x^{5/2}} \times \left(\frac{(-2\ln k_x)(1+k_x)}{k_x^{1/5}(1-k_x)}\right)^{5/4} \quad (82)$$

And, if to use (15), (74) and (82) we get the formula for the calculation of the effective dynamic viscosity module

$$E'' = \frac{15\pi m}{24k_p V_{0x}^{1/2} R^{1/2} \tau_x^{5/2}} \times (-2\ln k_x)^{5/4} \left(\frac{(1+k_x)}{k_x(1-k_x)}\right)^{1/4} \quad (83)$$

Obviously, if $k_x = 0$, then $E'' = 0$ too. We can find G' and G'' in the analogical way.

VISCOELASTIC STRESSES IN THE CONTACT AREA

Obviously that in the time of mutual approach between bodies, under action of the elastic forces, the instant volumetric elastic stresses arise in the deformation volume of the contact, and in the same time, the instant volumetric viscous stresses have place in process of the inside friction of the layered structures of a contact space between each other under action of the viscous forces. These can be found by using the classical expressions, as

$$\sigma_{cn} = \frac{d(\Delta V_{cx})}{dV_x}K', \quad \sigma_{bn} = \frac{d(\Delta V_{bx})}{dV_x}K'' \text{ and } \sigma_{c\tau} = \frac{d(\Delta V_{cy})}{dV_y}K', \quad \sigma_{bn} = \frac{d(\Delta V_{by})}{dV_y}K'' \qquad (84)$$

Where: σ_{cn} is the normal volumetric elastic stresses, σ_{bn} is the normal volumetric viscous stresses, $\sigma_{c\tau}$ is the tangential volumetric elastic stresses, $\sigma_{b\tau}$ is the tangential volumetric viscous stresses; ΔV_{cx} is the normal elastic deformed volume; ΔV_{bx} is the normal viscous deformed volume; ΔV_{cy} is the tangential elastic deformed volume; ΔV_{by} is the tangential viscous deformed volume; V_x is the normal deformation volume; V_y is the tangential deformation volume; K' is the elasticity bulk modulus; K'' is the viscosity bulk modulus.

The volumes of deformations in differential forms dV_{dx} - at the compressing and dV_{dy} - at the shift (see Fig. 2) are defined by formulas

$$dV_x = S_x dx, \quad dV_y = S_y dy, \qquad (85)$$

Where $S_x = \pi r^2 = k_p^2 Rx$ is the area of contact placed perpendicular to axle ***X***, S_y is the area of contact placed perpendicular to axle ***Y***. Formulas for S_y can be found by integration $dS_y = 2rdx$

$$S_y = 2\int_0^x rdx = 2k_p R^{1/2}\int_0^x x^{1/2}dx = \frac{4}{3}k_p R^{1/2}x^{3/2} \qquad (86)$$

and respectively we get

$$dV_x = k_p^2 Rxdx \text{ and } dV_y = \frac{4}{3}R^{1/2}x^{3/2}dy \qquad (87)$$

The deformed volumes, which are equal to the changing of volumes of deformations in the course of their deformations, can be defined according to the balance between the elementary works, which spent on deformations, and the elementary energies of these deformities:

$$\begin{cases} dA_{cx} = F_{cn}dx = \dfrac{4E'}{3}k_p R^{1/2}x^{3/2}dx = dW_x = d(\Delta V_{cx})k' \\ dA_{bx} = F_{bn}dx = \dfrac{4k_p R^{1/2}E''}{\omega_x K''}\dot{x}x^{1/2}dx = dW_{bx} = d(\Delta V_{bx})K'' \end{cases} \qquad (88)$$

and

$$\begin{cases} dA_{cy} = F_{c\tau}dy = G'P_x ydy = dW_{cy} = d(\Delta V_{cy})K' \\ dA_{by} = F_{b\tau}dy = \dfrac{G''}{\omega_y}P_x\dot{y}dy = dW_{by} = d(\Delta V_{by})K'' \end{cases} \qquad (89)$$

Where: dA_{cx} , dW_{cx} , dA_{cy} , dW_{cy} and dA_{bx} , dW_{bx} , dA_{by} , dW_{by} are the elementary work and elementary energy of elastic deformation at compression and shift.
Thus the deformed volumes we can write like the system four equations

$$\begin{cases} d(\Delta V_{cx}) = \dfrac{4E'}{3K'} k_p R^{1/2} x^{3/2} dx, \quad d(\Delta V_{bx}) = \dfrac{4k_p R^{1/2} E''}{\omega_x K''} \dot{x} x^{1/2} dx \\ \\ d(\Delta V_{cy}) = \dfrac{G'}{K'} P_x y dy , \quad d(\Delta V_{by}) = \dfrac{G''}{\omega_y K''} P_x \dot{y} dy , \end{cases} \tag{90}$$

The normal volumetric viscous and elastic contact stress according to the formulas (84), (87) and (90) can be written

$$\sigma_{cn} = \frac{4E'}{3\pi k_p} \frac{x^{1/2}}{R^{1/2}} , \quad \sigma_{bn} = \frac{4E''}{\pi \omega_x R^{1/2}} \times \frac{\dot{x}}{x^{1/2}} \tag{91}$$

Thus, for the normal volumetric viscoelastic is the sum the elastic and viscous normal stress

$$\sigma_n = \sigma_{cn} + \sigma_{bn} = \frac{4E'}{\pi k_p R^{1/2}} \left(\frac{x^{1/2}}{3} + \frac{\dot{x} tg\beta}{\omega_x x^{1/2}} \right) \tag{92}$$

On other hand the normal mean pressure P_m in the contact area, can be found as

$$P_m = \frac{F_n}{S_x} = \frac{4E'}{\pi k_p R^{1/2}} \left(\frac{x^{1/2}}{3} + \frac{\dot{x} tg\beta}{\omega_x x^{1/2}} \right), \tag{93}$$

thus as we can see $P_m = \sigma_n$.

The tangential volumetric viscous and elastic contact stress according to the formulas formulas (84), (87) and (90) can be written as

$$\sigma_{c\tau} = \frac{3G'}{4R^{1/2}} \frac{P_x y}{x^{3/2}} , \quad \sigma_{b\tau} = \frac{3G''}{4\omega_y R^{1/2}} \times \frac{P_x \dot{y}}{x^{3/2}} \tag{94}$$

And hence, the expression for the tangential volumetric viscoelastic stress can be written as

$$\sigma_\tau = \sigma_{c\tau} + \sigma_{b\tau} = \frac{3G' P_x}{4R^{1/2} x^{3/2}} \left(y + \frac{\dot{y} tg\beta}{\omega_y} \right) \tag{95}$$

Definition of the maximal stresses in the contact area

Most dangerous values of contact stresses have in the points when values of forces elasticity and viscosity maximal too. According to the hypothesis of maximum tangential stresses the equivalent stress can be defined under the formula

$$\sigma_e = \sqrt{\sigma_{nm}^2 + 4\sigma_{\tau m}^2} \tag{96}$$

Since as $\sigma_{nm} = P_c = \frac{3}{2}\sigma_n$ and $\sigma_{\tau m} = \sigma_\tau$, consequently we get

$$\sigma_e = \frac{6}{R^{1/2}}\sqrt{\left[\frac{E'}{\pi k_p}\left(\frac{x^{1/2}}{3} + \frac{\dot{x}tg\beta}{\omega_x x^{1/2}}\right)\right]^2 + \left[\frac{G'P_x}{4x^{3/2}}\left(y + \frac{\dot{y}tg\beta}{\omega_y x^{1/2}}\right)\right]^2} \quad (97)$$

Obviously, stresses reach maximum value σ_{em} in the case, when $t = \tau_1$, $x = x_m$ $\dot{x} = 0$ and $y = y_1$, and therefore, according to (97), the criterion of workability of contacting surfaces, when the maximal stress must not be higher than the endurance limit $\sigma_{\lim}$, can be written as

$$\sigma_{em} = \frac{6}{R^{1/2}}\sqrt{\left[\frac{E'}{\pi k_p}\left(\frac{x_m^{1/2}}{3}\right)\right]^2 + \left[\frac{G'P_{xm}}{4x_m^{3/2}}\left(y_1 + \frac{\dot{y}_1 tg\beta}{\omega_y x_m^{1/2}}\right)\right]^2} \le \sigma_{\lim} \quad (98)$$

Also, we can take in account that the dangerous value of stress takes place when the normal viscosity force has the extremum in some point of the time equal τ_b (see Fig.5.a). And as we know, the extremum of function has place in the point when its first derivative equal zero, therefore we can write

$$(F_{bn})' = \frac{4E''R^{1/2}}{\omega_x}(x^{1/2}\dot{x})' = 0 \quad (99)$$

So we receive the differential equation

$$2\ddot{x}x + \dot{x}\dot{x} = 0 \quad (100)$$

After the substitution of the functions $x(\tau_b)$, $\dot{x}(\tau_b)$ and $\ddot{x}(\tau_b)$ into (100) and then after a simplification we get the next equation for the calculation of the period of time τ_b

$$(3\delta_x^2 - 2\omega_x^2)tg^2(\omega_x\tau_b) - 6\omega_x\delta_x tg(\omega_x\tau_b) + \omega_x^2 = 0 \quad (101)$$

This algebraic equation very simple can be solved relative to the trigonometric function $tg(\omega_x\tau_b)$, and then the period of the time τ_b can be calculated. Then we can find x_b and the stress in the moment of the time τ_b by using (98).

ANALYSE AND CONCLUSION

It is a very important now to confirm of the correctness of the offered theories and methods, obtained in this article, if to compare them with others already available. For example, the equation for the elastic force $F_{cn} = \frac{4}{3}k_p E'R^{1/2}x^{3/2}$ have been obtained by the using of the "Method of the specific forces". In the case, when $k_p = 1$ we have the same solution like in the Hertz theory. Hence the Hertz theory gives the partial results in comparisons with proposed "MSF" method, because MSF" let to find the viscoelastic forces for any curvilinear contact between two surfaces, but Hertz case can be using only for the flat contact. The obtained result proves us that, the "Method of the specific forces" is definitely valid for finding of the normal elastic force, because if we know a functional dependency between *r* and *x*, we can always find the elastic force. It is obviously that, if it gives the correct way for the definition elastic force, and also as it was represented, it is valid for the definition of the viscous force and the tangential viscoelastic forces. We cannot find viscous force and the tangential viscoelastic forces by using the Hertz's theoretical model, but we can do this by using the "Method of the Specific Forces". It is obviously that, for the finding the normal viscous and the tangential viscoelastic forces, we can take $k_p = 1$, like according the Hertz theory, but we should be aware that, in this case, the contact area is a flat surface according to 2D tensor of deformations and h_x – the depth of indentation has to be equal to zero. But in reality, the contact surface takes the curvilinear shape, therefore, alternatively in this paper, the way of the finding of the radius of the contact area, by considering the geometry of contact between two curvilinear surfaces, have been proposed. It was received that the radius of contact area $r = f(x)$can be find by the equation $r^2 = 2Rx - x^2$. Since this equation is not convenient in using, and therefore it was proposed the finding ***r*** as $r^2 = k_p^2 Rx$, where $k_p = \sqrt{2 - \frac{x}{R}}$ is the correlation coefficient, which can be found by the method of iterations and consecutive approximations. If a deformation is small, when $R >> x$, hence we can take $k_p = \sqrt{2}$. And, if contact area is a flat, when $h_x = 0$, follows from Eq. 44* that $\left(\frac{R - D_2 R_2}{R_2} \right) = 0$ Practically the area of contact can be considered as a flat surface only in the case, when the surface of a semi-space in many times harder than the surface of a body. For example, it is possible in the case of impact between a rubber ball and a steel plane. Hence, it is obviously, that the "MSF" is the universal method, which can be used for any functional dependencies between the radius or the diameter of the contact area and the distance of the mutual approach (the total deformation) between two curvilinear surfaces. But nevertheless, we still have the question: What kind of the equation is better to take for finding of the radius of contact area, by the Hertz theory or directly by the way of consideration the geometry of the contact, like it is proposed in this article? Objectively to answer this question, we have to analyse simply logically the way as these equations were received. It was taken according to the Hertz theory that the contact surface is a flat, and the deformations are very small, the contact pressure is distributed analogically as an electrical potential (Remark: an electrical potential is the scalar function, but a pressure is the vector function), and then, on the basis of this main statements, the equation between the radius of the contact area and the normal elastic force, and the equation between the distance of the mutual approach and the normal elastic force as the effect have been obtained. Then only after that, in result of the comparison of these two equations by excluding the normal force (Landau, 1944, 1965), the expression $r^2 = Rx$ have been received. But in this article the analogical functional dependence have been proposed as the cause, in the result of the direct consideration the geometry of the contact. It was shown that, in the time of indentation of more hard surface into a soft surface, the contact surface takes a curvilinear elliptical shape (the function $r^2 = 2Rx - x^2$ is the elliptical function, which can be approximated by the parabolic function $r^2 = k_p^2 Rx$), where the point *B* (see Fig.4) is a special point where the deformations always equal zero, and the border of the area of contact always pass through this point *B*. Therefore, the proposed geometrical method is more exact, than according to the Hertz's theoretical model.

On the other hand, the equation for the normal viscous force $F_{bn} = 4k_p \frac{E''}{\omega_x} R^{1/2} \dot{x} x^{1/2}$ gives the similar result as it has place in the contact between two bodies with identical mechanical properties in the equation (1*), which have been obtained by Brilliantov, N. V., Spahn, F., Hertzsch, J.-M., and Poeschel, T. (1996). After the comparison of two these equations, since as $x = \xi$ and $R = R^{eff}$ we obtain the following

$$A = \frac{E'' k_p (1 - \nu^2)}{Y \omega_x} \qquad (102)$$

But since as $\omega_x = \frac{\pi}{\tau_x}$ and in the quasi-static conditions $E' = Y$ we get

$$A = \frac{k_p tg\beta}{\pi} \times \left(1 - \nu^2\right) \times \tau_x \qquad (103)$$

If $\beta = 0$ hence $A = 0$ too, it is a totally elastic impact. Thus we can find the parameter ***A*** by a very simple way using the "Method of the Specific Forces".

Also the equation (51) to determine the maximum displacement x_m have been derived. It is obvious, that in the case of k_x = 1 and k_p =1 we have the same result, as was obtained by Landau (1944,1965) for a totally elastic impact by using the Hertz Theory. It proves the correctness of the way of finding the Eq.65. But we have to understand that, this equation has the borders of application which can be found if to solve the next equation $\omega_x = \sqrt{\omega_{0x}^2 - \delta_x^2}$. First of all since as $\omega_{0x}^2 = \dfrac{C_x}{m}$ and $\delta_x = \dfrac{B_x}{2m}$, we can write that $\delta_x = \dfrac{3\omega_{0x}^2}{2\omega_x} tg\beta$ and we get the next algebraic equation $\omega_x^4 - \omega_{0x}^2\omega_x^2 + \omega_{0x}^4 tg^2\beta = 0$. This equation has only the one valid solution $\omega_x^2 = \dfrac{\omega_{0x}^2}{2}\left(1 + \sqrt{1 - 9tg^2\beta}\right)$ and it has the valid root only when $1 - 9tg^2\beta \geq 0$, therefore $tg\beta = \dfrac{E''}{E'} \leq \dfrac{1}{3}$, and according to Eq. 74 we get for a viscoelastic contact that

$$k_x \geq \frac{\pi - 1}{\pi + 1} \qquad (104)$$

In the case when $k_x \lhd \dfrac{\pi - 1}{\pi + 1}$ the plastic deformations will be have place in the zone of the contact.

In conclusion, first of all, let us to mark, that the method of specific viscoelastic forces allows to find the equations for all viscoelastic forces. The proposed method is a principally different with others in which are using the Hertz's theory, the classical theory of elasticity and the tensor algebra. In this method the new conception is proposed, how to find the elastic and viscous forces by an integration of the specific forces in the infinitesimal boundaries of the contact area. The radius of contact area can be taken according the Hertz theory or can be found by the considering the geometry of the contact. This method can be used in researches of the contact dynamics of any shape of contacting surfaces. Also in the article the method of the solution of the differential equations of a movement has been proposed and they have been solved. This method also can be used for determination of the dynamic mechanical properties of materials, and it can be used in the design of wear-resistant elements and coverings for components of machines and equipment, which are working in harsh conditions where they are subjected to the action of flow or jet abrasive particles. Also the theoretical and experimental statements which are presented here can be useful in the design of elements and details machines and mechanism which are being in the conditions of the dynamic contact. The results of the experimental and theoretical research and the method of the specific forces presented in this article can be used for the determination of the viscoelastic forces, contact stresses, durability and fatigue life for a wide spectrum of the tasks relevant to collisions between solid bodies under different loading conditions. Opportunities exist to use the obtained results practically in the design and development of new advanced materials, wear-resistant elastic coatings and elements for pneumatic and hydraulic systems, stop valves, fans, centrifugal pumps, injectors, valves, gate valves and in other installations. Also the using of this theory gives an opportunity for the development of analytical and experimental methods allowing optimising the basic dynamic and mechanical visco-elastic qualities already existing materials and in the development new advanced materials and elements of machines. Also this theory can be used not only for visco-elastic contact and also for any other kind of contacts, such as the elasto- plastic contact and for the elasto-visco-plastic contact too.

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